# Rational maps

$Y, Y'$ irreducible varieties.

rational map $\phi : Y \dashrightarrow Y'$

$(\underset{\text{open}}{U \subset Y}, \underset{\text{morphism}}{\phi : U \to Y})_\sim$ $\quad (U_1 \subset Y, \phi_1 : U_1 \to Y) \sim (U_2, \phi_2 : U_2 \to Y)$ if $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$

- $\exists$ representative with maximal domain of definition $U_\phi$, $Y \setminus U_\phi =: Z_\phi$
- can compose dominant rational maps $\rightsquigarrow$ category.
- $\phi : Y \dashrightarrow Y'$ birational if $U \xrightarrow[\cong]{\phi} \phi(U)$ for some $U \subset Y$ open
  or, equiv, if $\phi$ dominant and $\phi^\flat : K(Y') \xrightarrow{\cong} K(Y)$

**Lemma** $Y$ normal, $Y'$ proper.
$\phi : Y \dashrightarrow Y'$ rational map
$\Rightarrow$ $Z_\phi = Y \setminus U_\phi$ has codim 2.

$\Rightarrow$ if $\dim(Y) = 2$ then $Z_\phi$ finite dim

**Proof** let $p \in Y$ s.t. $\dim(\mathcal{O}_{Y,p}) = 1$
$Y$ normal $\Rightarrow$ $\mathcal{O}_{Y,p}$ DVR
valuative criterion for properness
$\Rightarrow$ $\phi$ extends uniquely to $p$ $\square$

- $Z_\phi$ codim 2 $\Rightarrow$ $\phi^* : \mathrm{Pic}(Y') \to \mathrm{Pic}(U_\phi) \cong \mathrm{Pic}(Y)$.

**Example** $X$ surface, $p \in X$

$\beta : \mathrm{Bl}_p X \to X$ is birational.
defined everywhere
$\beta^{-1} : X \dashrightarrow \mathrm{Bl}_p X$ defined on $X \setminus p = U_{\beta^{-1}}$
but not at $p \in Z_{\beta^{-1}}$.

- $X$ surface. $\phi : X \dashrightarrow Y$ rational map.
  - $Z_\phi$ ~~is~~ zero-dim.
- $C \subset X$ irred. define $\phi(C) = \overline{\phi(C \setminus Z_\phi)}$ "strict transform"

Theorem $\phi : Y \dashrightarrow Y'$ rational map
$Y'$ projective, $Y$ normal.
Then $\exists Z \subset Y$ ~~...~~ and $\tilde{\phi} : Bl_Z Y \to Y'$ s.t.

$$\begin{tikzcd} Bl_Z Y \arrow[r, "\tilde{\phi}"] \arrow[d] & Y' \\ Y \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

commutes.

Proof sketch: wlog. $Y' = \mathbb{P}^m$

$\Rightarrow$ $\phi$ defined by $(\mathcal{L}, s_0, \dots, s_m)$

with $V(s_0, \dots, s_m) = Z_\phi$ codim 2.

$s_0, \dots, s_m$ vanish at $Z_\phi$

$\Rightarrow$ $\beta^*_{\beta} s_i$ vanish at $E_{Z_\phi} Y$ and define sections $\tilde{s}_i \in H^0(\beta^* \mathcal{L}(-E_{Z_\phi} Y))$

$\tilde{\phi}$ corresponds to $(\beta^* \mathcal{L}(-E_{Z_\phi} Y), \tilde{s}_0, \dots, \tilde{s}_m)$ $\square$

Example $\phi : \mathbb{A}^2 \dashrightarrow \mathbb{P}^1$
$(x,y) \longmapsto [x:y]$ "line through the origin and $(x,y)$"

$$\begin{tikzcd} \tilde{\phi} : Bl_0 \mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1 \arrow[r, "pr_2"] \arrow[dr] & \mathbb{P}^1 \\ & \mathbb{A}^2 \arrow[u, dashed] \end{tikzcd}$$

$\circlearrowleft$

**Theorem** $X$ surface, $Y$ irred proj var.
$\phi : X \dashrightarrow Y$ rational map
$\Rightarrow \exists$ finite sequence of blow ups at (reduced) points $\tilde{X} := X_n \xrightarrow{\beta_n} \cdots \to X_1 \xrightarrow{\beta_1} X$ and a morphism $\pi : \tilde{X} \to Y$ s.t.

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\beta_1 \circ \cdots \circ \beta_n"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}$$

commutes.

**Proof** wlog $Y = \mathbb{P}^m$ and $\phi(X)$ not contained in a hyperplane.

$\Rightarrow \phi$ defined by a line bundle with sections
$\mathcal{L}$, $s_0, \dots, s_m \in H^0(\mathcal{L})$ s.t. $V(s_0, \dots, s_m)$ zero-dim'l

if $V(s_0, \dots, s_m) = \emptyset$, done.
else let $x_1 \in |V(s_0, \dots, s_m)|$ and consider blow up

$$X_1 = Bl_{x_1}(X) \xrightarrow{\beta_1} X \dashrightarrow \mathbb{P}^m, \quad E_1 = \beta_1^{-1}(x_1)$$

$k_1 = \min\{\mathrm{ord}_{E_1}(\beta_1^* s_0), \dots, \mathrm{ord}_{E_1}(\beta_1^* s_m)\}$

$\Rightarrow$ get sections $s_i^{(1)} := \beta_1^* s_i(-k_1 E) \in H^0(\underbrace{\beta_1^* \mathcal{L}(-k_1 E)}_{=: \mathcal{L}_1})$

which agree with $s_i$ on $X_1 \setminus E_1 = X \setminus x$
and $V(\underline{s}^{(1)})$ is zero dimensional b/c
$E_1 \not\subseteq V(\underline{s}^{(1)})$ (and $E_1$ is the only curve in $X_1$ that could have made $V(\underline{s}^{(1)})$ positive dim'l)

repeat as long as $V(\underline{s}^{(\ell)}) \neq \emptyset$.
Why does the process terminate?

$$(\mathcal{L}_\ell \cdot \mathcal{L}_\ell) = (\beta_\ell^* \mathcal{L}_{\ell-1} - k_\ell E_\ell, \beta^* \mathcal{L}_{\ell-1} - k_\ell E_\ell)$$
$$= (\mathcal{L}_{\ell-1} \cdot \mathcal{L}_{\ell-1}) - k_\ell^2 < (\mathcal{L}_{\ell-1} \cdot \mathcal{L}_{\ell-1})$$

$(\mathcal{L}_\ell \cdot \mathcal{L}_\ell) \geq 0$ b/c $V(\underline{s}^{(\ell)})$ zero dimensional.
can replace a copy of $\mathcal{L}_\ell$ with one represented by sum of irred curves which don't share components w/ $\mathcal{L}_\ell$. $\square$

<u>Lemma</u> $X'$ irred. proj. var. dim(2), possibly singular.
$\pi : X' \to X$ birational morphism to $X$ surface
if $\pi^{-1}$ undefined at $p \in X$ (i.e., $p \in Z_{\pi^{-1}} \subseteq X$)
then $\pi^{-1}(p) \subseteq X'$ is a curve.

<u>Proof</u> Consider $\phi : X \overset{\pi^{-1}}{\dashrightarrow} X' \hookrightarrow \mathbb{P}^m$

Suppose $\pi^{-1}(p)$ finite dimensional.
Then there is a hyperplane $H \subset \mathbb{P}^n$ s.t.
$H \cap \pi^{-1}(p) = \emptyset$

$\Rightarrow p \notin V(\phi^* x_0, \dots, \phi^* x_n)$ $\qquad \phi^* x_i \in H^0(\phi^* \mathcal{O}_{\mathbb{P}^n}(H))$

$\Rightarrow \phi$ defined at $p$

$\Rightarrow \pi^{-1}$ defined at $p$ ↯ $\square$

<u>Lemma</u> $\phi : X \dashrightarrow X'$ birational map of surfaces.
Suppose $\phi^{-1}$ not defined at $p'$. then
there is a curve $C \subseteq X$ s.t. $\phi(C) = p'$.

<u>Proof</u>

$$\begin{tikzcd} & X \times X' \supset \overline{X} := \overline{\mathrm{graph}(\phi)} & \\ & U_\phi \times X' \supset \mathrm{graph}(\phi) \arrow[u, phantom, "\cup"] \arrow[dl] \arrow[dr] & \\ X \supset U_\phi \arrow[rr, "\phi"] & & X' \end{tikzcd}$$

$\overline{X}$ irred. proj. var. dim(2) and
$$\begin{tikzcd} & \overline{X} \arrow[dl, "q"'] \arrow[dr, "q'"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$
are birational

$\phi^{-1}$ undefined at $p'$ $\Rightarrow (q')^{-1}$ undefined at $p'$

lemma $\Rightarrow \exists\, C \subset \overline{X}$ s.t. $q'(C) = p'$

$\Rightarrow q(C) \subseteq X$ curve with $\phi(q(C)) = p'$ $\square$

<u>Prop</u> $\pi : X' \to X$ birational morphism of surfaces.
Suppose $\pi^{-1}$ undefined at $p \in X$.
Then we have a commutative diagram of birational morphisms.

$$\begin{tikzcd} X' \arrow[r, "\tilde{\pi}"] \arrow[dr, "\pi"'] & \mathrm{Bl}_p X =: \tilde{X} \supset E = \beta^{-1}(p) \arrow[d, "\beta"] \arrow[dl, phantom, "///"] \\ & X \end{tikzcd}$$

<u>Proof</u> ~~Claim~~ We show that the birational map $\tilde{\pi} : X' \xrightarrow{\pi} X \overset{\beta^{-1}}{\dashrightarrow} \mathrm{Bl}_p X$
is a morphism (i.e. defined everywhere)

Suppose $\tilde{\pi}$ not defined at $p' \in X$.
$\Rightarrow \tilde{\pi}(p') = p$. and $\exists\, \tilde{C} \subset \tilde{X}$ curve s.t.
$\tilde{\pi}^{-1}(\tilde{C}) = p'$
$\Rightarrow \beta(\tilde{C}) = p \Rightarrow \tilde{C} \subseteq E \Rightarrow \tilde{C} = E$
$\tilde{X} \supset Z_{\tilde{\pi}^{-1}}$ zero dimensional $\Rightarrow \exists\, e \in E$ s.t. $\tilde{\pi}^{-1}$ defined at $e$

$\Rightarrow \tilde{\pi}^{-1}(e) = p'$

let $x, y$ be local coord. at $p \in X$.
locally over $p$ $\mathrm{Bl}_p X$ defined by equation
$\{x\tilde{y} - y\tilde{x} = 0\} \subseteq \mathbb{P}^1_V$ wlog $e = [1:0] = \infty$
$\Rightarrow$ local coordinates $x, \frac{\tilde{y}}{\tilde{x}}$ at $e$.
and $E$ cut out by $x$.

$\pi^{-1}$ undefined at $p \Rightarrow \exists\, C' \subset \pi^{-1}(p)$ s.t. $\pi(C') = p$
say $f \in \mathcal{O}_{X',p'}$ local eqn cutting out $C'$
$\pi^{-1}(p) = V(\pi^\flat(x), \pi^\flat(y)) \Rightarrow \pi^\flat(x), \pi^\flat(y) \in (f) \subseteq \mathcal{O}_{X',p'}$
but $\mathcal{O}_{X',p'} \subseteq \mathcal{O}_{\tilde{X},e}$, $x$ local coord @ $e$
$\Rightarrow x \notin \mathfrak{m}^2_{\tilde{X},e} \Rightarrow x \notin \mathfrak{m}^2_{X',p'} \Rightarrow \frac{x}{f} \in \mathcal{O}_{X',p'}$ $\longrightarrow$

$$\Rightarrow \frac{\tilde{y}}{\tilde{x}} = \left(\frac{y}{f}\right)\left(\frac{x}{f}\right)^{-1} \quad \text{in } \mathcal{O}_{X',p'}$$

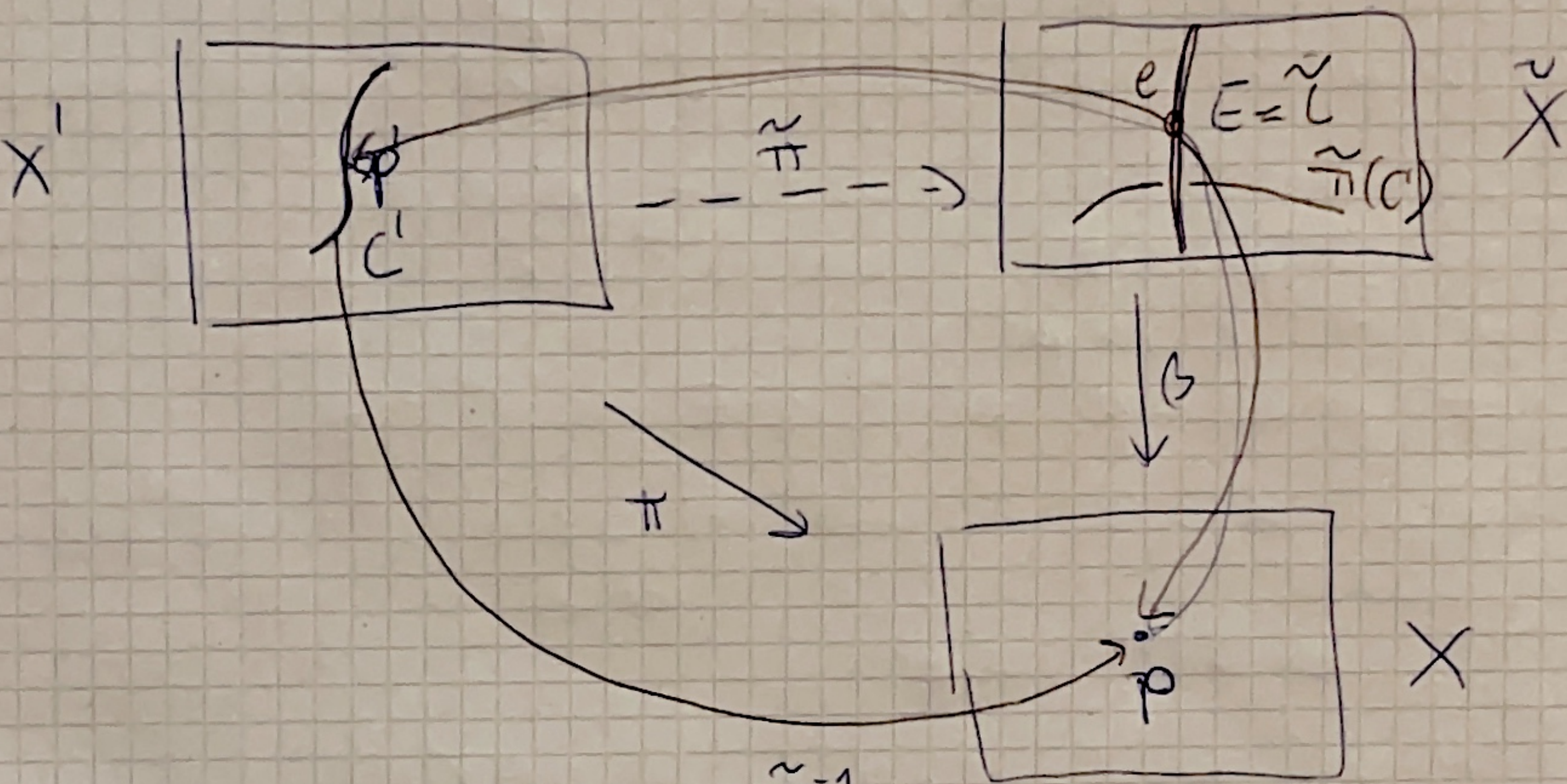
$$\Rightarrow \frac{\tilde{y}}{\tilde{x}} \in \mathfrak{m}_{X',p'}$$

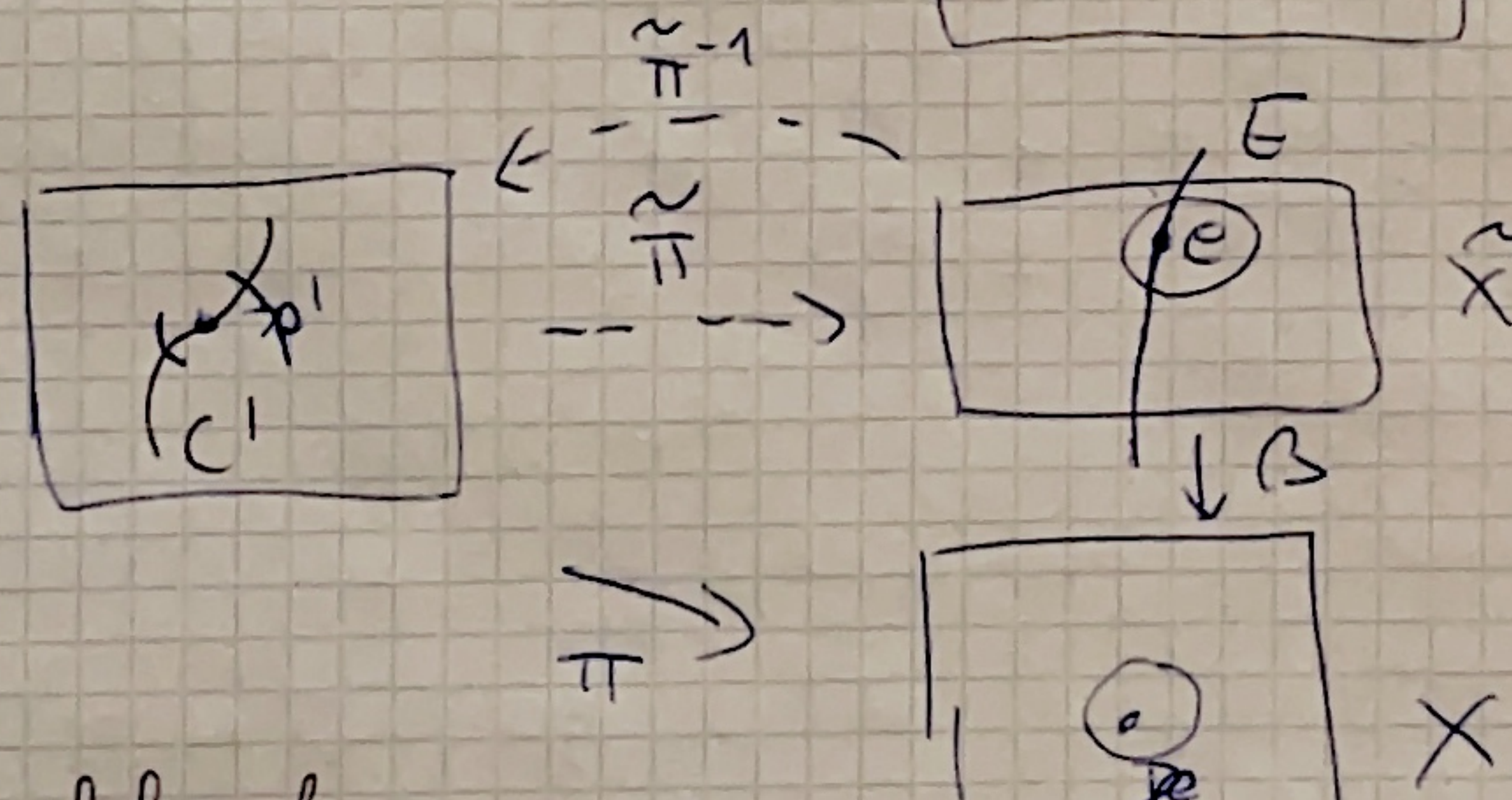
$$\tilde{\pi}^{-1}(E) = x' \Rightarrow \forall g \in \mathfrak{m}_{X',p'}:$$

$$\Rightarrow g \in \mathcal{O}_{\tilde{X},e} \Rightarrow g \in (x)$$

$$\Rightarrow \frac{\tilde{y}}{\tilde{x}} \in (x) \ \lightning$$

$\square$




Since $\tilde{\pi}^{-1}$ defined
@ e ~~ta~~ expect
local parameters at e
to be transformed into local parameters at p'
but can't be because the local parameters must both
~~functions which~~ vanish on C!

<u>Theorem</u> $\pi: X' \to X$ birational morphism of surfaces

$\Rightarrow \exists$ sequence of blow ups at points

$$X_n \xrightarrow{\beta_n} X_{n-1} \to \dots \xrightarrow{\beta_1} X_1 \to X$$

and iso $\eta: X' \to X_n$ s.t. $\pi = \beta_1 \circ \dots \circ \beta_n \circ \eta$.

<u>Proof.</u> if $\pi$ iso, then done.

else $\exists\, p_1 \in X$ s.t. $\pi^{-1}$ undefined at $p_1$.

$\Rightarrow \pi$ factors through blow up at $p_1$

$$\pi = \beta_1 \circ \pi_1$$

if $\pi_1$ iso done, else repeat.

Why terminates?

$n(\pi_i) = \#\{\text{irred curves } C \subset X' \text{ contracted by } \pi_i\}$

drops at every step b/c

for some $C_i \subset X'$ w/ $\pi_{i-1}(C_i) = p_{i-1}$

have $\pi_i(C_i) = E_i$.

but $0 \leq n(\pi_i)$ $\Rightarrow$ terminates. $\square$

<u>Corollary</u> $\phi: X \dashrightarrow X'$ birational map of surfaces

$\Rightarrow \exists$ comm. diag.

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

where $\pi, \pi'$ are compositions of blow ups at points and isomorphisms.